A TECHNIQUE FOR PLANE-SYMMETRIC SOUND FIELD ANALYSIS IN THE FAST MULTIPOLE BOUNDARY ELEMENT METHOD

Y. YASUDA
Institute of Industrial Science, The University of Tokyo
4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan
yyasuda@iis.u-tokyo.ac.jp

T. SAKUMA
Institute of Environmental Studies, The University of Tokyo
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
sakuma@k.u-tokyo.ac.jp

Received 16 November 2003
Revised 30 June 2004

The fast multipole boundary element method (FMBEM) is an advanced BEM, with which both the operation count and the memory requirements are \( O(N^a \log^b N) \) for large-scale problems, where \( N \) is the degree of freedom (DOF), \( a \geq 1 \) and \( b \geq 0 \). In this paper, an efficient technique for analyses of plane-symmetric sound fields in the acoustic FMBEM is proposed. Half-space sound fields where an infinite rigid plane exists are typical cases of these fields. When one plane of symmetry is assumed, the number of elements and cells required for the FMBEM with this technique are half of those for the FMBEM used in a naive manner. In consequence, this technique reduces both the computational complexity and the memory requirements for the FMBEM almost by half. The technique is validated with respect to accuracy and efficiency through numerical study.

Keywords: Fast multipole algorithm; hierarchical cell structure; plane-symmetric field; half-space field; boundary element method; Helmholtz equation.

1. Introduction

The conventional boundary element method (BEM) requires large amount of computational time and memory requirements, when the degree of freedom (DOF) for an analyzed problem is large. The operation count of the BEM is \( O(N^3) \) with direct solvers, or \( O(N^2) \) with appropriate iterative solvers, and the memory requirements are \( O(N^2) \), where \( N \) is the degree of freedom (DOF), because of its dense matrices. To overcome this shortcoming, the fast multipole BEM (FMBEM) has been developed in various fields,\(^1\)–\(^4\) including the field of acoustics.\(^5\)–\(^14\) This is an advanced BEM with the use of the fast multipole algorithm (FMA), which was a fast algorithm originally developed by Rokhlin\(^15\) for large-scale problems. Both
the operation count and the memory requirements of the FMBEM are $O(N^a \log^b N)$, where $a \geq 1$ and $b \geq 0$. The values $a$ and $b$ depend on details of the algorithm.

The FMBEM has large advantage over the conventional BEM in efficiency, whereas some numerical techniques that can be used in the conventional BEM cannot be directly used in the FMBEM. One of these techniques is that for plane-symmetric sound fields. Examples of these fields are half-space fields, where an infinite rigid plane exists: sound fields around noise barriers on the ground,\textsuperscript{16–18} sound fields around risers on a stage in a hall\textsuperscript{19} etc. In these fields, the infinite rigid plane can be replaced by mirror images of objects with respect to the plane. Other examples are symmetrical rooms, objects, periodic walls and diffusers.

For analyses of plane-symmetric sound fields, simple change in the Green’s function reduces the computational costs in the conventional BEM.\textsuperscript{16–19} With one plane of symmetry, for example, the computational complexity becomes about $(1/2)^3$ with a direct solver and about $(1/2)^2$ with an iterative solver, and the memory requirements about $(1/2)^2$, because the number of equations for the linear system are reduced by half. On the other hand, similar change in the Green’s function is not possible in the FMBEM, since the way to evaluate interactions between elements is completely different between these methods; in the FMBEM, hierarchical cell structure is employed, and interactions between elements are evaluated by calculating interactions between cells, using multipole expansion. However, efficient calculation for plane-symmetric sound fields with the FMBEM is possibly realized by making good use of the hierarchical cell structure: by arranging hierarchical cell structure symmetrically corresponding to the symmetry of objects and using the symmetry in calculating coefficients of cells.

In this paper, we present a technique for plane-symmetric sound fields in the FMBEM. In the below, we call the FMBEM applied with this technique the symmetrical FMBEM. This technique reduces the number of equations for the linear system by half, when one plane of symmetry is assumed. In consequence, both the computational complexity and the memory requirements for the symmetrical FMBEM are about $1/2$ as small as those for the FMBEM used in a naive manner. Section 2 briefly shows the calculation process of the FMBEM. In Sec. 3, concrete computational procedures of the symmetrical FMBEM are described. Here three formulations of the FMBEM are considered, i.e. singular (basic form: BF) and hypersingular (normal derivative form: NDF) formulation, and combined formulation of the two sets of equations to avoid the fictitious eigenfrequency difficulties with external problems (Burton–Miller formulation\textsuperscript{20}). In Sec. 4, numerical results with the symmetrical FMBEM are presented and compared to those with the FMBEM used in a naive manner, and validate the accuracy and the efficiency of the symmetrical FMBEM.

2. Calculation of FMBEM

Here we briefly describe the outline of the FMBEM and its computational procedures. For more details of the FMBEM, see the Ref. 10. Another brief description of the FMBEM can be seen in the Ref. 21. Throughout this section, time convention $\exp(-j\omega t)$ is used.
2.1. Outline of BEM

In the field satisfying the three-dimensional Helmholtz equation, the sound pressure on the smooth boundary $\Gamma$ is described using the Kirchhoff–Helmholtz integral equation. By discretizing the equation with three kinds of boundary conditions, a rigid boundary $\Gamma_0$, a vibration boundary $\Gamma_1$, and an absorption boundary $\Gamma_2$, the following system of equations, called basic form (BF), is obtained:

$$(E + B + C) \cdot p = j\omega \rho A \cdot v,$$  \hspace{1cm} (2.1)

where $p$ is the vector of the sound pressure $p$ on the boundary (unknown), $v$ is the vector of the normal component of the surface velocity $v$ (given), $\omega$ is the angular frequency, $\rho$ is the air density, and the entries of the influence coefficient matrices are represented by

$$E_{ij} = -\frac{1}{2} \delta_{ij},$$  \hspace{1cm} (2.2)

$$A_{ij} = a_j(r_i) = \int_{\Gamma_1} N_j(r_q) G(r_i, r_q) dS_q,$$  \hspace{1cm} (2.3)

$$B_{ij} = b_j(r_i) = \int_{\Gamma} N_j(r_q) \frac{\partial G(r_i, r_q)}{\partial n_q} dS_q,$$  \hspace{1cm} (2.4)

$$C_{ij} = c_j(r_i) = jk \int_{\Gamma_2} N_j(r_q) \frac{G(r_i, r_q)}{z(r_q)} dS_q,$$  \hspace{1cm} (2.5)

where $\delta$ is Kronecker’s delta, $r_i$ is the position vector of the $i$th node, $k$ is the wave number, $z$ is the acoustic impedance ratio, $\partial/\partial n_q$ denotes the normal derivative, and $N_j$ is the interpolation function of the $j$th node. $G$ is the Green’s function given by

$$G(r_p, r_q) = \frac{\exp(jkr_{pq})}{4\pi r_{pq}},$$  \hspace{1cm} (2.6)

where $r_{pq} = |r_p - r_q|$ is the distance between points $p$ and $q$.

In a similar way, the following system of equations in the normal derivative form (NDF), which is based on the derivative of the Helmholtz–Kirchhoff integral equation with respect to the normal on the boundary, is obtained as

$$(G + H + J) \cdot p = j\omega \rho (F + I) \cdot v,$$  \hspace{1cm} (2.7)

where

$$I_{ij} = \frac{1}{2} \delta_{ij} \bigg|_{\Gamma_1},$$  \hspace{1cm} (2.8)

$$J_{ij} = \frac{jk}{2z_j} \delta_{ij} \bigg|_{\Gamma_2},$$  \hspace{1cm} (2.9)

$$F_{ij} = f_j(r_i) = \int_{\Gamma_1} N_j(r_q) \frac{\partial G(r_i, r_q)}{\partial n_i} dS_q,$$  \hspace{1cm} (2.10)

$$G_{ij} = g_j(r_i) = \int_{\Gamma} N_j(r_q) \frac{\partial^2 G(r_i, r_q)}{\partial n_i \partial n_q} dS_q,$$  \hspace{1cm} (2.11)
As for Burton–Miller formulation, Eqs. (2.1) and (2.7) are simply combined with a combination factor.

2.2. Outline of FMBEM

When solving Eq. (2.1) or Eq. (2.7) with an iterative solver, the FMBEM efficiently achieves matrix-vector multiplications by applying multipole expansion in multiple levels using hierarchical cell structure. Since it is not necessary to keep matrices themselves, the memory requirements also drastically decrease. The following briefly shows the outline and the computational process of the FMBEM.

2.2.1. On hierarchical cell structure

Figure 1 shows an example of boundary and hierarchical cell structure in two dimensions. A cube (a square in two dimensions) circumscribing the whole boundary is determined as a root cell, which is divided into eight child cubes (level 1). Each divided cube is also divided in turn (level 2, 3, ..., L). Only the cubes including nodes are called cells.

2.2.2. Evaluation of matrix-vector products using FMA

According to the multipole translation theory with plane wave expansion,\textsuperscript{6–8} the Green’s function Eq. (2.6) can be transformed into the following expression, which corresponds to

\[
H_{ij} = h_j(r_i) = j k \int_{\Gamma_j} N_j(r_q) \frac{1}{z(r_q)} \frac{\partial G(r_i, r_q)}{\partial n_i} dS_q.
\]  

(2.12)

Fig. 1. Two-dimensional diagram of hierarchical cell structure (the lowest level number \( L = 4 \)) and boundary, with illustration of three paths for evaluation of influence from point \( q \) to \( p \). Diagram of steps 1 to 5 in the FMBEM is also illustrated.
the computational procedures of steps 1 to 5 described in the latter part of the section:

\[
G(r_p, r_q) = \frac{jk}{16\pi^2} \oint E_p\lambda_{m_L}(k) \prod_{l=1}^{L-1} E_{\lambda_{m_{l+1}}\lambda_{m_l}}(k) \\
\cdot \frac{(\beta_{\lambda_{m_{l'}}\lambda_{m_l}'}(k) + \gamma_{\lambda_{m_{l'}}\lambda_{m_l}'}(k))p_j}{\alpha_{\lambda_{m_{l'}}\lambda_{m_l}'}(k)v_j} d\hat{k},
\]

where

\[
T_{\lambda_{m_{l'}}\lambda_{m_l}'}(k) \prod_{l=1}^{L-1} E_{\lambda_{m_{l+1}}\lambda_{m_l}}(k) E_{\lambda_{m_{l'}}\lambda_{m_l}}(k) d\hat{k},
\]

and

\[
G(r_p, r_q) = \frac{k^2}{16\pi^2} \oint (n_i \cdot \hat{k}) E_i\lambda_{m_L}(k) \prod_{l=1}^{L-1} E_{\lambda_{m_{l+1}}\lambda_{m_l}}(k) \\
\cdot \frac{(\beta_{\lambda_{m_{l'}}\lambda_{m_l}'}(k) + \gamma_{\lambda_{m_{l'}}\lambda_{m_l}'}(k))p_j}{\alpha_{\lambda_{m_{l'}}\lambda_{m_l}'}(k)v_j} d\hat{k},
\]

where

\[
\alpha_{\lambda_{m_{l'}}\lambda_{m_l}'}(k) = \int_{\Gamma_1} N_j(r_q) E_{\lambda_{m_{l'}}\lambda_{m_l}'}(k) dS_q,
\]

\[
\beta_{\lambda_{m_{l'}}\lambda_{m_l}'}(k) = jk \int_{\Gamma_1} N_j(r_q) E_{\lambda_{m_{l'}}\lambda_{m_l}'}(k)(n_q \cdot \hat{k}) dS_q,
\]

\[
\gamma_{\lambda_{m_{l'}}\lambda_{m_l}'}(k) = jk \int_{\Gamma_2} N_j(r_q) E_{\lambda_{m_{l'}}\lambda_{m_l}'}(k) \frac{E_{\lambda_{m_{l'}}\lambda_{m_l}'}(q)}{z(r_q)} dS_q.
\]

In the procedures for computation, the integral \( \oint d\hat{k} \) is calculated numerically, using the next equation:

\[
\oint f(\hat{k}) d\hat{k} = \sum_{n=1}^{K} w_n f(\hat{k}_n),
\]

where \( w_n \) and \( \hat{k}_n \) are the weights and nodes, respectively.
where \( w_n \) are the weights for the quadrature points \( \hat{k}_n \), and \( K \) is the number of the quadrature points. Contributions from/to cells are evaluated using coefficients at the quadrature points \( \hat{k}_n \) of the integral in steps 1 to 5 described below. These coefficients are called outgoing, interaction, and incoming coefficients \( \xi, \tau, \) and \( \zeta \).

### 2.2.3. Procedures for computation

We explain the concrete procedures for calculation of matrix-vector products for three formulations, BF, NDF, and Burton–Miller formulation. The procedures consist of six steps.

**Step 1.** Compute the outgoing coefficients \( \xi_{m_L} \) of each cell at each quadrature point \( \hat{k}_n^L \) at the lowest level \( L \), by

\[
\begin{bmatrix}
\xi^p_{m_L}(\hat{k}_n^L) \\
\xi^v_{m_L}(\hat{k}_n^L)
\end{bmatrix}
= \sum_{j \in G_{m_L}} \begin{bmatrix}
(\beta\lambda_{m_L,j}(\hat{k}_n^L) + \gamma\lambda_{m_L,j}(\hat{k}_n^L))p_j \\
\alpha\lambda_{m_L,j}(\hat{k}_n^L)v_j
\end{bmatrix},
\tag{2.22}
\]

where the superposed \( p \) and \( v \) denote the coefficients for sound pressure and for surface velocity, respectively, and \( G_{m_L} \) denotes the set of elements in cell \( m_L \).

**Step 2.** Compute the outgoing coefficients \( \xi_{m_l} \) of each cell at each quadrature point \( \hat{k}_n^{l+1} \) at the next higher level, with interpolation for quadrature points, by

\[
\begin{bmatrix}
\xi^p_{m_l}(\hat{k}_n^{l+1}) \\
\xi^v_{m_l}(\hat{k}_n^{l+1})
\end{bmatrix}
= \sum_{m_{l+1} \in C_{m_l}} \sum_{n=1}^{K_{l+1}} E_{\lambda_{m_l}\lambda_{m_{l+1}}}(\hat{k}_n^{l}) \sum_{n=1}^{K_{l+1}} W_{n} W_{n'} \begin{bmatrix}
\xi^p_{m_{l+1}}(\hat{k}_n^{l+1}) \\
\xi^v_{m_{l+1}}(\hat{k}_n^{l+1})
\end{bmatrix},
\tag{2.23}
\]

where \( K_l \) is the number of the quadrature points for the spherical integral at level \( l \), \( W_{n'} \) are the interpolation coefficients, and \( C_{m_l} \) denotes the child cell set, which consists of child cells for \( m_l \). This computation is executed in the upward order at each level \( (l = L - 1, L - 2, \ldots, 2) \).

**Step 3.** Compute the interaction coefficients \( \tau_{m_l} \) of each cell at each quadrature point \( \hat{k}_n^{l+1} \) at each level \( (l = 2, 3, \ldots, L) \), by

\[
\begin{bmatrix}
\tau^p_{m_l}(\hat{k}_n^{l+1}) \\
\tau^v_{m_l}(\hat{k}_n^{l+1})
\end{bmatrix}
= \sum_{m_l' \in T_{m_l}} T_{\lambda_{m_l}\lambda_{m_l'}}(\hat{k}_n^{l+1}) \begin{bmatrix}
\xi^p_{m_l'}(\hat{k}_n^{l+1}) \\
\xi^v_{m_l'}(\hat{k}_n^{l+1})
\end{bmatrix},
\tag{2.24}
\]

where \( T_{m_l} \) denotes the interaction cell set, which consists of the cells which are not neighbors of \( m_l \) but whose parents are neighbors of parent cell of \( m_l \). Figure 2(b) shows an example of an interaction cell set in two-dimensions.

**Step 4.** Compute the incoming coefficients \( \zeta_{m_{l+1}} \) of each cell at each quadrature point \( \hat{k}_n^{l+1} \) at the next lower level, with adjoint interpolation\(^2\) for quadrature points, by

\[
\begin{bmatrix}
\zeta^p_{m_{l+1}}(\hat{k}_n^{l+1}) \\
\zeta^v_{m_{l+1}}(\hat{k}_n^{l+1})
\end{bmatrix}
= \sum_{n'=1}^{K_{l+1}} \frac{W_{n'}^{l+1}}{W_n} E_{\lambda_{m_l+1}\lambda_{m_l}}(\hat{k}_n^{l}) \begin{bmatrix}
\zeta^p_{m_l}(\hat{k}_n^{l+1}) + \tau^p_{m_l}(\hat{k}_n^{l+1}) \\
\zeta^v_{m_l}(\hat{k}_n^{l+1}) + \tau^v_{m_l}(\hat{k}_n^{l+1})
\end{bmatrix},
\tag{2.25}
\]

where \( \zeta_{m_2}(\hat{k}_n^{l+1}) = 0 \) if \( l = 2 \). This computation is executed in the downward order at each level \( (l = 2, 3, \ldots, L - 1) \).
Step 5. Compute the far influence on each node, $\phi_{F,i}$ (for the BF) or $\psi_{F,i}$ (for the NDF), at the lowest level $L$, by

$$
\begin{bmatrix}
\phi_{F,i}^P \\
\phi_{F,i}^V
\end{bmatrix} = \frac{jk}{16\pi^2} \sum_{n=1}^{K_L} u_n^L E_{i\lambda m_L}(k_n^L) \begin{bmatrix}
\zeta_{m_L}(k_n^L) + \tau_{m_L}(k_n^L) \\
\zeta_{m_L}(k_n^L) + \tau_{m_L}(k_n^L)
\end{bmatrix},
$$

$$
\begin{bmatrix}
\psi_{F,i}^P \\
\psi_{F,i}^V
\end{bmatrix} = \frac{-k^2}{16\pi^2} \sum_{n=1}^{K_L} u_n^L (n_i \cdot k_n^L) E_{i\lambda m_L}(k_n^L) \begin{bmatrix}
\zeta_{m_L}(k_n^L) + \tau_{m_L}(k_n^L) \\
\zeta_{m_L}(k_n^L) + \tau_{m_L}(k_n^L)
\end{bmatrix}.
$$

Regarding Burton–Miller formulation, a similar evaluation to Eqs. (2.26) and (2.27) is possible by

$$
\begin{bmatrix}
\phi_{F,i}^{P} + \alpha \psi_{F,i}^{P} \\
\phi_{F,i}^{V} + \alpha \psi_{F,i}^{V}
\end{bmatrix} = \frac{jk}{16\pi^2} \sum_{n=1}^{K_L} u_n^L (1 + (n_i \cdot k_n^L)) E_{i\lambda m_L}(k_n^L) \begin{bmatrix}
\zeta_{m_L}(k_n^L) + \tau_{m_L}(k_n^L) \\
\zeta_{m_L}(k_n^L) + \tau_{m_L}(k_n^L)
\end{bmatrix},
$$

where the combination factor $\alpha = 1/jk$ is assumed.

Step 6. Compute the near influence on each node, $\phi_{N,i}$ (for the BF) or $\psi_{N,i}$ (for the NDF), as the total effect of the elements in the neighbor cell set at the lowest level $L$, by

$$
\begin{bmatrix}
\phi_{N,i}^P \\
\phi_{N,i}^V
\end{bmatrix} = \sum_{m_i' \in \mathcal{N}_{m_i}} \sum_{j \in \mathcal{G}_{m_i'}} \begin{bmatrix}
(E_{ij} + B_{ij} + C_{ij})p_j \\
A_{ij}v_j
\end{bmatrix},
$$

$$
\begin{bmatrix}
\psi_{N,i}^P \\
\psi_{N,i}^V
\end{bmatrix} = \sum_{m_i' \in \mathcal{N}_{m_i}} \sum_{j \in \mathcal{G}_{m_i'}} \begin{bmatrix}
(G_{ij} + H_{ij} + J_{ij})p_j \\
(F_{ij} + I_{ij})v_j
\end{bmatrix},
$$

where $\mathcal{N}_{m_i}$ denotes the neighbor cell set, which consists of $m_i'$ itself and neighboring cells of $m_i'$. Figure 2(a) shows an example of an neighbor cell set in two-dimensions. Regarding Burton–Miller formulation, the above two equations are simply combined with the combination factor.
Finally, compute the complete influence on each node by adding the far and the near influences, which gives the matrix-vector products in Eqs. (2.1) and (2.7).

3. Calculation of Symmetrical FMBEM

Here we describe the FMBEM with an efficient technique for plane-symmetric sound fields (symmetrical FMBEM). As mentioned above, both the computational complexity and the memory requirements for the FMBEM are about 1/2 with this technique as small as those for the FMBEM used in a naive manner. This technique is applicable to plane-symmetric sound fields with two or three planes of symmetry that are orthogonal one another. In these cases, the computational complexity and the memory requirements become about 1/4 or 1/8, respectively. We describe the numerical procedures for the symmetrical FMBEM, and theoretically estimate the effect of reduction of the computational complexity and the memory requirements.

3.1. Numerical method

3.1.1. Evaluation of image region

Figure 3 shows an example of a plane-symmetric object and hierarchical cell structure. Hierarchical cell structure is arranged symmetrically corresponding to the symmetry of the object. Here we define an unit part of the symmetrical shape, which consists of parts of the analyzed object and of the hierarchical cell structure, as an unit region for calculation, and the other region as an image region. Cells at lower levels are produced only in the unit region (above half region in Fig. 3). As shown in Fig. 4, where \( \langle x \rangle \) denotes mirror image of \( x \) with respect to the plane of symmetry, relation of positions among cells \( m_l, m_{l+1} \) and

![Diagram](image.png)

Fig. 3. Diagram of the FMBEM for a plane-symmetric sound field (symmetrical FMBEM).
a quadrature point for integration of unit sphere $\langle \hat{k}_n^l \rangle$ is the same as relation among $\langle m_l \rangle$, $\langle m_{l+1} \rangle$ and $\langle \hat{k}_n^l \rangle$. Therefore, an equation for outgoing coefficients $\xi_{\langle m_l \rangle}(\hat{k}_n^l)$ is obtained as

$$\xi_{\langle m_l \rangle}(\hat{k}_n^l) = \xi_{\langle m_{l+1} \rangle}(\hat{k}_n^l).$$

(3.31)

This relation is satisfied also for interaction and incoming coefficients $\tau$ and $\zeta$. One can obtain the values of these coefficients of cells in image regions from values in the unit region using Eq. (3.31). Thus, calculation only in the unit region is sufficient in steps 1 and 2, where contributions from elements or cells at a level are accumulated to their parent cells, and in steps 4 and 5, where contributions to cells at a level are translated into contributions to their child cells or their inside elements. As for step 3, where contributions from interaction cell sets are evaluated, and step 6, where contributions from elements in neighbor cell sets are evaluated with the conventional BEM, it is necessary to consider the contributions from cells in the image region, only when a cell in the unit region is located near the plane of symmetry, as shown in Fig. 3. Specifically, in step 3, when parts of interaction cells of a cell in the unit region are located in the image region, contributions from the image cells are obtained using Eq. (3.31). In step 6, when parts of neighboring cells of a cell in the unit region are located in the image region, contributions from elements in the image cells are computed with the conventional BEM.

3.1.2. Procedures for computation

Computational procedures of matrix-vector products of the symmetrical FMBEM are described. These procedures basically correspond to the six steps of the FMBEM described in the previous section. We deal with three BEM formulations: BF, NDF, and Burton–Miller formulation. Procedures for steps 1, 2, 4, and 5 are the same as those in the previous section,
except for the point that these procedures are applied only to the unit region for calculation in the symmetrical FMBEM. Thus, we describe procedures only for steps 3 and 6.

**Step 3.** Compute the interaction coefficients \( \tau_{m_l} \) of each cell at each quadrature point \( \hat{k}_{nl}^l \) at each level \( (l = 2, 3, \ldots, L) \), by

\[
\begin{bmatrix}
\tau_{m_l}^p(\hat{k}_{nl}^l) \\
\tau_{m_l}^v(\hat{k}_{nl}^l)
\end{bmatrix} = \sum_{m_l' \in T_{m_l}} T_{\lambda_{m_l} m_l'} \begin{bmatrix}
\xi_{m_l'}^p(\hat{k}_{nl}^l) \\
\xi_{m_l'}^v(\hat{k}_{nl}^l)
\end{bmatrix} + \sum_{\langle m'_l \rangle \in T_{m_l}} T_{\lambda_{m_l} \langle m'_l \rangle} \begin{bmatrix}
\xi_{m'_l}^p(\hat{k}_{nl}^l) \\
\xi_{m'_l}^v(\hat{k}_{nl}^l)
\end{bmatrix}.
\]

(3.32)

The second term of the right-hand side expresses the contribution from image cells of \( m'_l \) \((m'_l \) belongs to the unit region). This second term shows that one can use the existent values of \( T_{LM} \) and \( \xi \) in the unit region to calculate the contribution from the image region. This term is calculated only if the parent cell of \( m_l \) is on the plane of symmetry, as shown in Fig. 5. Equation (3.32) is used for all three formulations, BF, NDF and Burton–Miller formulation.

**Step 6.** Compute the near influence on each node, \( \phi_{N,i} \) (for the BF) or \( \psi_{N,i} \) (for the NDF), as the total effect of the elements in the neighbor cell set at the lowest level \( L \), by

\[
\begin{bmatrix}
\phi_{N,i}^p \\
\phi_{N,i}^v
\end{bmatrix} = \sum_{m'_L \in N_{m_L}} \sum_{j \in G_{m'_L}} \begin{bmatrix}
(E_{ij} + B_{ij} + C_{ij})p_j \\
A_{ij}v_j
\end{bmatrix} + \sum_{\langle m'_L \rangle \in N_{m_L}} \sum_{j \in G_{m'_L}} \begin{bmatrix}
(E_{ij} + B_{ij} + C_{ij})p_j \\
A_{ij}v_j
\end{bmatrix},
\]

(3.33)

Fig. 5. Diagram of step 3 in the symmetrical FMBEM: (a) one of the cases where the parent cell of \( m_l \) is on the plane of symmetry and the second term of the right-hand side of Eq. (3.32) is calculated, and (b) one of the other cases where the second term is not calculated.
Fig. 6. Diagram of step 6 in the symmetrical FMBEM: (a) one of the cases where $m_L$ is on the plane of symmetry and the second terms of the right-hand side of Eqs. (3.33) and (3.34) are calculated, and (b) one of the other cases where the second terms are not calculated.

\[
\begin{bmatrix}
\psi_{p,N,i}' \\
\psi_{v,N,i}'
\end{bmatrix} = \sum_{m'_L \in N, m_L} \sum_{j \in G_{m'_L}} \left[ (G_{ij} + H_{ij} + J_{ij}) p_j ight] \\
+ \sum_{(m'_L) \in N, m_L} \sum_{j \in G_{m'_L}} \left[ (G_{i(jj)} + H_{i(jj)} + J_{i(jj)}) p_j ight].
\] (3.34)

The second term of the right-hand side expresses the contribution from image cells of $m'_L$ ($m'_L$ belongs to the unit region), and is calculated only if cell $m_L$ is on the plane of symmetry, as shown in Fig. 6. As for Burton–Miller formulation, the above two equations are simply combined with a combination factor, thus, the way to evaluate contributions from the image region is the same as Eqs. (3.33) and (3.34).

### 3.2. Computational efficiency

#### 3.2.1. Computational complexity

The main process of the FMBEM consists of the setup process and the iterative process. The former is the process for calculation of coefficients that are not necessary to iteratively calculate. The latter is the process where the six steps described above are iteratively executed to calculate matrix-vector products. Here we discuss the computational complexity for both processes of the symmetrical FMBEM.

In the setup process, the coefficients $\alpha, \beta, \gamma$ in step 1, the translation coefficients $T_{LM}$ in step 3, and the influence coefficients for near interaction $E_{ij} + B_{ij} + C_{ij}$ and $A_{ij}$ (for BF), or $G_{ij} + H_{ij} + J_{ij}$ and $F_{ij} + I_{ij}$ (for NDF) in step 6 are calculated. The computational complexity for $\alpha, \beta, \gamma$ and the influence coefficients for near interaction is $1/2$ in the symmetrical FMBEM compared to the FMBEM in a naive manner, since the number of elements in the symmetrical FMBEM is reduced by half. On the other hand, the complexity for $T_{LM}$ does
not decrease. This is because $T_{LM}$ are calculated at each level using common interaction cell set $T'_l$, instead of using relation of positions between each cell and its interaction cell set, and the number of common interaction cells at each level does not decrease in the symmetrical FMBEM. We have described in detail the calculation of $T_{LM}$ using common interaction cell set $T'_l$ in Ref. 21.

In the iterative process, the number of coefficients obtained in each step are reduced by half in the symmetrical FMBEM compared to the FMBEM used in a naive manner, since coefficients only in the unit region are required in the symmetrical FMBEM. Therefore, the complexity for the iterative process is completely $1/2$ in all six steps.

Consequently, the total computational complexity with the symmetrical FMBEM is reduced almost by half compared with the FMBEM in a naive manner. However, the improvement of the computational complexity by this technique is a little spoiled when the proportion of the computational complexity for $T_{LM}$ to all complexity is large. This spoilage does not occur unless the number of iteration in the iterative process is quite small, and the proportion of the complexity for $T_{LM}$ to the total complexity for the setup process is large. The latter case can occur when problems with 1D-shaped objects, such as long ducts and noise barriers for road or railroad traffic, are analyzed without any special settings of hierarchical cell structure, because the large size of the root cell required for 1D-shaped objects cause the large amount of the computational complexity for $T_{LM}$. However, this problem can be avoided by adopting an appropriate setting of hierarchical cell structure.$^{21}$

3.2.2. Memory requirements

The memory requirements, completely similar to the computational complexity for the setup process, are $1/2$ as much as those of the FMBEM in a naive manner, except for the memory for $T_{LM}$. When analyzing 1D-shaped problems, it is better to adopt an appropriate setting of hierarchical cell structure$^{21}$ to avoid increase of the memory for $T_{LM}$.

4. Numerical Results

We execute a numerical study to validate the symmetrical FMBEM with respect to accuracy and efficiency. Figure 7 shows an interior sound field in a rigid cube $d$ m wide, with a point source located at the center of the cube. Quadrature constant elements with width of less than $1/8$ of the wavelength are used for mesh generation, and Bi-CGStab$^{24}$ without preconditioning is used as an iterative solver for linear systems. The numerical items for calculations are determined identical with those in Ref. 11. The computation is executed with the supercomputer HITACHI SR8000. One plane of symmetry parallel to sides of the cube is used for analysis with the symmetrical FMBEM. Table 1 shows other conditions for calculation.

Figure 8 shows an example of computational results of sound pressure level distribution on the floor, using the FMBEM in a naive manner (conventional FMBEM), and sound pressure level difference between the conventional and the symmetrical FMBEM. The difference
Fig. 7. Geometry of an analyzed model. A point source is located at the center of the cube. All boundaries are rigid. One plane of symmetry is used for analysis with the symmetrical FMBEM.

Table 1. Computational efficiency for analyzing sound fields in a rigid cube $d$ m wide, with a point source at the center, using the FMBEM in a naive manner (conventional FMBEM) and using the symmetrical FMBEM. $N$ is the degree of freedom (DOF), and $L$ is the lowest level number of hierarchical cell structure.

<table>
<thead>
<tr>
<th>$kd$</th>
<th>Type of FMBEM</th>
<th>$N$</th>
<th>$L$</th>
<th>Iteration</th>
<th>Time [sec]</th>
<th>Memory [MB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.14</td>
<td>conventional</td>
<td>1,536</td>
<td>2</td>
<td>5</td>
<td>28</td>
<td>26.6</td>
</tr>
<tr>
<td>9.14</td>
<td>symmetrical</td>
<td>768</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>14.4</td>
</tr>
<tr>
<td>73.12</td>
<td>conventional</td>
<td>98,304</td>
<td>5</td>
<td>59</td>
<td>24,235</td>
<td>1,486.8</td>
</tr>
<tr>
<td>73.12</td>
<td>symmetrical</td>
<td>49,152</td>
<td>5</td>
<td>58</td>
<td>11,848</td>
<td>799.6</td>
</tr>
</tbody>
</table>

Fig. 8. Distribution of sound pressure level on the floor, obtained with the FMBEM in a naive manner (conventional FMBEM) and with the symmetrical FMBEM, and sound pressure level difference between these two methods, at $kd = 73.12$. 
Table 2. Differences between results with the FMBEM in a naive manner (conventional FMBEM) and with the symmetrical FMBEM, averaged over all nodes on boundaries of the cube. $\varepsilon_{\text{sym}}$ is defined as Eq. (4.35). $N$ is the degree of freedom (DOF), and $L$ is the lowest level number of hierarchical cell structure.

<table>
<thead>
<tr>
<th>$kd$</th>
<th>Conventional</th>
<th>Symmetrical</th>
<th>$L$</th>
<th>$10 \log_{10} \varepsilon_{\text{sym}}$ [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.14</td>
<td>1,536</td>
<td>768</td>
<td>2</td>
<td>$-127.6$</td>
</tr>
<tr>
<td>73.12</td>
<td>98,304</td>
<td>49,152</td>
<td>5</td>
<td>$-34.1$</td>
</tr>
</tbody>
</table>

is less than 0.05 dB at almost all nodes except for the dip positions, and the maximum value of the difference is less than 0.3 dB. Table 2 shows differences between results with the conventional and the symmetrical FMBEM, averaged over all nodes on the boundary. To avoid problematic influence of spatial sampling of node positions, here the mean difference $\varepsilon_{\text{sym}}$ is defined as

$$
\varepsilon_{\text{sym}} = \frac{\sum_n |p_{\text{con}}(r_n)|^2 - |p_{\text{sym}}(r_n)|^2|}{\sum_n |p_{\text{con}}(r_n)|^2},
$$

(4.35)

where $p_{\text{con}}(r_n)$ and $p_{\text{sym}}(r_n)$ are sound pressures at a node $r_n$ obtained with the conventional and the symmetrical FMBEM, respectively. It is seen that the computational accuracy by the symmetrical FMBEM is almost the same as that by the conventional FMBEM, especially with small DOF. The symmetrical FMBEM has no factors making accuracy worse than the conventional FMBEM. Therefore, the difference between these methods is probably attributed to the difference of linear systems: the system obtained with the conventional FMBEM has twice as many equations as the symmetrical FMBEM with one plane of symmetry. The results obtained with the two systems are the same theoretically, though different numerically due to round-off errors and a stopping criterion for an iterative solver, usually close to but unequal to 0.

Table 1 shows the computational efficiency and the conditions for calculation using the conventional and the symmetrical FMBEM. The symmetrical FMBEM reduces both of the computational time and the memory requirements almost by half, not quite dependent on DOF.

5. Concluding Remarks

An efficient technique for analysis of plane-symmetric sound fields in the fast multipole boundary element method has been proposed. Concrete computational procedures of the FMBEM used with this technique were described for three formulations: basic form, normal derivative form, and Burton-Miller formulation to avoid the fictitious eigenfrequency difficulties with external problems. Numerical results showed that both the computational
complexity and the memory requirements for the FMBEM with this technique were about 1/2 as small as those for the FMBEM used in a naive manner, when one plane of symmetry was assumed. This technique can be expanded for plane-symmetric sound fields with two or three planes of symmetry that are orthogonal one another; in these cases, the computational complexity and the memory requirements become about 1/4 or 1/8, respectively. This technique is applicable not only to sound fields with real objects of symmetrical shapes, but also to half-space fields where an infinite rigid plane exists.

References